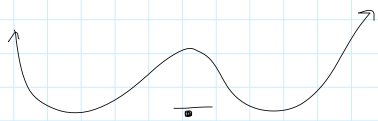
Integrals cont

Saturday, March 20, 2021 11:35 AM



skip 3.4, 3.7 in [Co]

responsible for: 3.1, 3.2, 3.3, 3.5, 3.6 in [Co]

Lagrange multiplier is solving for $\frac{d}{dt} = 0$

Suppose we integrate $\iiint_R f(x,y)\,dx\,dy$ where R is a region $\mathbb{R}^2$, f defined on R.

A partition P of R is a decomposition
$R = R_1 \cup R_2 \cup \cdots \cup R_N$
whose interiors don't intersect.

mesh (P) = max diameter (R_i)
→ Diameter $(R_i) = \sup_{\vec{v}_1, \vec{v}_2 \in R_i} \|\vec{v}_1 - \vec{v}_2\|$

diameter of a rectangle = length of diagonal
" " " $\triangle$ = " " longest side

Q: why diameter, not area, for mesh?
A. Consider $N \times \frac{1}{N^2}$ rectangle
↳ Area really small $\left(\frac{1}{N}\right)$
↳ Diameter: large
$= \sqrt{N^2 + \frac{1}{N^4}} \simeq N$

We want to avoid such an R_i
↳ want each R_i to be small in all directions

Now

$$\iint_R f(x,y)\,dx\,dy = \lim_{\text{mesh} \to 0} \sum_{i=1}^{N} f(x_i, y_i)\,\text{area}(R_i)$$
$(x_i, y_i) \in R_i$

Single integrals
replace area w/ length
triple integrals
replace area w/ volume

call area(R_i) "=" $\Delta A = \Delta$ area
↳ for triple integrals, $\Delta V = \Delta$volume

Q/ What is $dx\,dy$?
A/ it's like $\Delta x \Delta y$ = area of $\Delta y \boxed{}_{\Delta x}$
= Δ area

eg if $R = [a,b] \times [c,d]$, divide into $N = M^2$ little rectangles, all congruent to each other.
Then each rectangle is Δx by Δy
$\Delta x = \frac{b-a}{M}$ $\quad \Delta y = \frac{d-c}{M}$
↳ in 3D

$$\Delta x = \frac{b-a}{M} \qquad \Delta y = \frac{d-c}{M}$$

↳ in 3D

$$\Delta V = \Delta x \Delta y \Delta z$$

Let's explain change of variables formula using this idea

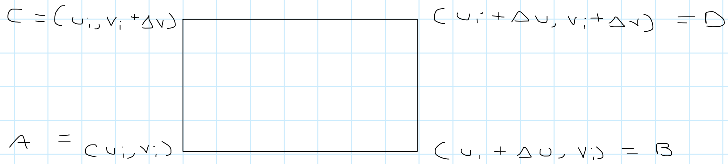
Suppose we have coords u,v and x,y and a coordinate transformation

$x = g_1(u,v)$ $\qquad g = (g_1, g_2)$

$y = g_2(u,v)$ $\qquad (x,y) = g(u,v)$

Q/ How to relate $dxdy$ to $dudv$?

A/ suppose we have a little rectangle in coords u,v like so:



area of this rectangle (in u,v coords) is $\Delta u \Delta v$ "=" $du dv$

Q/ If we apply g to this rectangle, what (approximately) is the area is xy-coords of the resulting shape?

approximation should better as $\Delta u, \Delta v \to 0$

set $(x_i, y_i) = g(u_i, v_i) = g(A)$

$g(B) = g(u_i + \Delta u, v_i)$

gets better as $\Delta u \to 0$

$$\approx g(u_i, v_i) + \Delta u \cdot \frac{\partial g}{\partial u}(u_i, v_i)$$

$$= (x_i, y_i) + \Delta u \left(\frac{\partial g_1}{\partial u}(u_i, v_i), \frac{\partial g_2}{\partial u}(u_i, v_i) \right)$$

$g(C) = g(u_i, v_i + \Delta v)$

$$\approx g(u_i, v_i) + \Delta v \frac{\partial g}{\partial v}(u_i, v_i)$$

$$= (x_i, y_i) + \Delta v \left(\frac{\partial g_1}{\partial v}(u_i, v_i), \frac{\partial g_2}{\partial v}(u_i, v_i) \right)$$

$g(D) = g(u_i + \Delta u, v_i + \Delta v)$

$$= (x_i, y_i) + \Delta u \frac{\partial g}{\partial u}(u_i, v_i) + \Delta v \frac{\partial g}{\partial v}(u_i, v_i)$$

→ So, g applied to the rectangle has verticies

$g(A), g(B), g(C), g(D)$

$$\approx (x_i, y_i), (x_i, y_i) + \vec{r}_1, (x_i, y_i) + \vec{r}_2, (x_i, y_i) + \vec{r}_1 + \vec{r}_2$$

where

$$\vec{r}_1 = \Delta u \frac{\partial g}{\partial u}(u_i, v_i) = \Delta u \left(\frac{\partial g_1}{\partial u}(u_i, v_i), \frac{\partial g_2}{\partial u}(u_i, v_i) \right)$$

$$\vec{r}_2 = \Delta v \frac{\partial g}{\partial v}(u_i, v_i) = \Delta v \left(\frac{\partial g_1}{\partial v}(u_i, v_i), \frac{\partial g_2}{\partial v}(u_i, v_i) \right)$$

Note by Thm 1.B in [Co], the area of this parallelogram is $\|\vec{r}_1 \times \vec{r}_2\|$

$$= \left\| \left(0, 0, \Delta u \frac{\partial g_1}{\partial u} \Delta v \frac{\partial g_2}{\partial v} - \Delta u \frac{\partial g_2}{\partial v} \Delta v \frac{\partial g_1}{\partial v}\right) \right\|$$

$$= \left| \Delta \frac{\partial g_1}{\partial u} \Delta \frac{\partial g_2}{\partial v} - \Delta u \frac{\partial g_2}{\partial u} \Delta v \frac{\partial g_1}{\partial v} \right|$$

$$= \Delta u \Delta v \left| \frac{\partial g_1}{\partial u} \frac{\partial g_2}{\partial v} - \frac{\partial g_2}{\partial u} \frac{\partial g_1}{\partial v} \right|$$

this is the area in the xy plane

$$d\left(\substack{\text{area in} \\ \text{xy plane}}\right) = dx\,dy$$
$$= du\,dv \left| \frac{\partial g_1}{\partial u} \frac{\partial g_2}{\partial v} - \frac{\partial g_2}{\partial u} \frac{\partial g_1}{\partial v} \right|$$

this is the change of variables formula

Note $dx\,dy$ really means $d(\text{Area})$ where area is taken in $\underbrace{\text{xy coordinates}}_{\text{like choosing a unit of measure for area (like } m^2 \text{ vs } ft^2)}$

This formula relates area in xy-coords in uv-coords

Q/ What about 3 dim?

A/ Use a 3×3 determinant = volume of a parallelpiped

To compute w/change of variables in 3-dim, use formula in book.

Below here is NOT IN SCOPE

<u>Remark</u> determinants in general are the scaling factor for n-volume

1-volume = length 3-volume = volume

2-volume = area 4-volume = hypervolume

<u>Remark</u> What if we want to use the determinant instead of its absolute value?

signed area vs area

signed area = $\pm$ area

and it's $(-)$ if opposite orientation

Note:

If using signed area, must keep track of the order of x and y.

For us:

$$\iint f\,dx\,dy = \iint f\,dy\,dx$$

For signed area

$$\iint f\,dx \wedge dy \quad \text{and} \quad dy \wedge dx = -dx \wedge dy$$

Why is it useful?

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$\Rightarrow dx \wedge dy =$$

$$\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)\wedge\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\right)$$

$$=\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}du\wedge dv+\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}dv\wedge du+(\text{stuff})\,du\wedge du+(\text{stuff})\,dv\wedge dv$$

Note

1. $du\wedge du=-du\wedge du$

   $\Rightarrow du\wedge du = dv\wedge dv = 0$

2. $dv\wedge du=-du\wedge dv$

Now no absolute value, these $\wedge$ have to do with exterior powers & differential forms